Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
International Journal of Heat and Mass Transfer 48 (2005) 2110–2116
International Journal of HEAT and MASS TRANSFER
www.elsevier.com/locate/ijhmt

Technical Note

Remarkable improvement of the Lévêque solution for isoflux heating with a combination of the transversal method of lines (TMOL) and a computer-extended Fröbenius power series

Antonio Campo [a,*], Cristina H. Amon [b]

[a] Department of Mechanical Engineering, The University of Vermont, Burlington, VT 05405, United States
[b] Department of Mechanical Engineering, Carnegie Mellon University, Pittsburgh, PA 15213, United States

Received 13 October 2003; received in revised form 13 January 2005

Abstract

This paper addresses the second Lévêque problem with uniform wall heat flux adhering to the original two-dimensional energy conservation equation with variable coefficients in cylindrical coordinates. The semi-analytic procedure to be proposed combines the transversal method of lines (TMOL) with the Fröbenius version of the power series method. The hybrid solution that emerges from this combination holds unique features that distinguish it from the traditional solution methods. In principle, due to the presence of a two-point backward formulation, the approximate analytic solution is considered first-order accurate. However, using as guidance the computed local convection coefficient at various transversal lines, it is demonstrated that the approximate analytic TMOL/Fröbenius solution is better than first-order accurate.

1. Introduction

The incipient thermal development of fully established laminar flows with viscous fluids when moving inside round tubes has been recognized as the Lévêque problem in the literature on convective heat transfer (Shah and London [1], Gnielinski [2] and Hewitt [3]). As a collation, the complete thermal development of fully established laminar flows with viscous fluids in round tubes is named the Graetz/Nusselt problem [1–3]. Back in 1928, Lévêque [4] conceived an elegant, closed-form solution for the temperature distribution for the case of a hot fluid flowing in a round tube with a fully developed laminar velocity and prescribed wall temperature (a Dirichlet boundary condition). Lévêque exploited the fluid physics in depth by inferring that the hydrodynamic laminar boundary layer flow dominated the core flow in the immediate entrance of the temperature development region. Lévêque's idea had its foundation on two main assumptions: (a) a geometric assumption that replaced the round tube with a "flat plate passage" and (b) a hydrodynamic assumption in which the fluid velocity varied linearly with the transverse coordinate for short distances from $x = 0$. Consequently, the thermal boundary layer and the related

* Corresponding author. Tel.: +1 802 656 3320; fax: +1 802 656 8802.
E-mail address: acampo@emba.uvm.edu (A. Campo).

0017-9310/$ - see front matter © 2005 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ijheatmasstransfer.2005.01.008

Nomenclature

c_p	specific isochoric heat capacity	x	axial coordinate
D	tube diameter	X	dimensionless x, $x/RRePr$
h	local convection coefficient		
k	thermal conductivity	*Greek symbols*	
Nu	local Nusselt number, hD/k	η	dimensionless r, r/R
q_w	wall heat flux	ϕ	dimensionless T, $(T - T_e)/[(q_w R)/k]$
r	radial coordinate	μ	viscosity
R	tube radius	ρ	density
Re	Reynolds number, $\rho\bar{u}D/\mu$		
Pr	Prandtl number, $\mu c_p/k$	*Subscripts*	
T	temperature	b	mean bulk
u	axial velocity	e	entrance
$\bar{u}$	mean of u	w	wall

heat penetration in the viscous fluid were restricted to a thin shell of fluid adjacent to the tube wall, but near the entrance.

From a strict mathematical standpoint, what Lévêque did was to rescale the variable coefficient caused by the parabolic velocity profile in the two-dimensional energy equation expressed in cylindrical coordinates. Thereafter, with an appended Couette-type linear velocity, he solved the simplified two-dimensional energy equation in rectangular coordinates by means of the similarity transformation technique. As remarked by Kevorkian [5], Lévêque provided a singular perturbation solution of the two-dimensional energy equation in cylindrical coordinates near the singularity at the early beginning of the heat exchange region.

The natural extension of the Lévêque problem pertains to the replacement of an isothermal-walled tube (a Dirichlet boundary condition) by an isoflux-walled tube (a Neumann boundary condition) [1]. This gave rise to the so-called second Lévêque problem, which was treated by Bird et al. [6] many years later resorting to a variant of the singular perturbation solution. The present paper addresses the second Lévêque problem treated in [6], but from a radically different perspective focusing on mathematical concepts solely. In actuality, the applicable two-dimensional energy equation in cylindrical coordinates is employed here without making any a priori physical assumptions about the shape of the velocity profile and the duct configuration, like Lévêque did. In this regard, a hybrid computational procedure known as the transversal method of lines (TMOL) is envisioned as capable of transforming the 2-D partial differential energy equation with variable coefficients into an adjoint ordinary differential energy equation with identical variable coefficients. Due to the presence of variable coefficients and a regular singular point, the ordinary differential energy equation is solved with the power series method, in particular the Fröbenius method. In compliance with a tight convergence criterion imposed, a computer-extended Fröbenius series was used to generate a succession of semi-analytical radial temperature profiles at fixed axial stations $T(\Delta x, r)$ close to the origin $x \to 0$.

Arguably, when round tubes are heated by uniform heat flux, the wall temperature $T_w = T(\Delta x, R)$ and not the local convection coefficient h at Δx, is the ultimate target quantity that needs to be determined for purposes of thermal engineering design. Thus, the goodness of the wall temperatures T_w is discussed at length in the last section of the paper dedicated to the presentation of results. In synthesis, the findings of this paper provided an alternate prediction method for solving the second Graetz/Nusselt problem in round tubes with uniform isoflux walls with a reasonable margin of error. Literally, the present method turns out to be very simple and its implementation is straightforward.

2. Mathematical formulation

Consider the heating of a viscous Newtonian fluid flowing laminarly through a round tube with a uniform heat flux, wherein at the entrance of a heat exchange region $x = 0$ the velocity is fully established and the temperature is uniform. For a constant-property fluid, the growth of the thermal boundary layer is described by the dimensionless two-dimensional energy equation in cylindrical coordinates

$$(1 - \eta^2)\frac{\partial \phi}{\partial X} = \frac{\partial^2 \phi}{\partial \eta^2} + \frac{1}{\eta}\frac{\partial \phi}{\partial \eta} \tag{1}$$

and subject to the set of boundary conditions

$$\phi = 0, \quad X = 0 \tag{2a}$$

$$\frac{\partial \phi}{\partial \eta} = 0, \quad \eta = 0 \tag{2b}$$

$$\frac{\partial\phi}{\partial\eta} = 1, \quad \eta = 1 \tag{2c}$$

The standard dimensionless variables in the formulation are $\phi = (T - T_e)/[(q_w R)/k]$ for the temperature T, $X = x/RRePr$ for the axial coordinate x and $\eta = r/R$ for the radial coordinate r.

2.1. Thermal parameters of interest

Once the dimensionless temperature field $\phi(X,\eta)$ is determined, the mean bulk temperature is obtained from the integral

$$\phi_b(X) = 4\int_0^1 \phi(X,\eta)(1-\eta^2)\eta\,d\eta \tag{3}$$

However, whenever uniform heating is applied at a tube wall, fluid physics dictates that the variation the mean bulk temperature can be established by a simple energy balance (see Appendix A). The end product is the mean bulk temperature rising linearly with the axial coordinate as follows

$$\phi_b(X) = 4X \tag{4}$$

In the traditional analysis of convection heat transfer, the key unknown is the local convection coefficient

$$h = \frac{q_w}{T_w - T_b} \tag{5a}$$

This ratio is usually channeled through the local Nusselt number Nu:

$$Nu = \frac{hD}{k} = \frac{2}{\phi_w - \phi_b} \tag{5b}$$

where $\phi_w = \phi(X,1)$ denotes the dimensionless wall tube temperature.

3. Semi-analytic solution

3.1. The transversal method of lines (TMOL)

We apply first a hybrid computational procedure, such as the transversal method of lines (TMOL), also known as Rothe method in the mathematical literature (Rothe [7]). Accordingly, the first-order axial derivative in Eq. (1) is replaced by a backward finite-difference formulation at a station ΔX measured from the entrance $X = 0$, whereas the first- and second-order radial derivatives remain continuous. Then, insertion of the entrance boundary condition of Eq. (2a) meaning null ϕ, yields the adjoint differential-difference energy equation

$$\frac{d^2\phi}{d\eta^2} + \frac{1}{\eta}\frac{d\phi}{d\eta} - \frac{1}{\Delta X}(1-\eta^2)\phi = 0 \tag{6}$$

The domain of η is $[0,1]$ paired with the first transversal line placed at ΔX. In Eq. (6), the embedded parameter ΔX specifies an axial stretching interval. Further, the corresponding radial boundary conditions acting at the transversal line ΔX are rewritten as

$$\frac{d\phi}{d\eta} = 0, \quad \eta = 0 \tag{7a}$$

$$\frac{d\phi}{d\eta} = 1, \quad \eta = 1 \tag{7b}$$

Fundamentally, the validity of the above boundary value problem, recounted by Eqs. (6) and (7) is limited to the vicinity of the origin, i.e., for $X \to 0$, because the backward finite-difference formulation participating in Eq. (6) has an error of order ΔX (Bender and Orszag [8]). In principle, the solution of the boundary value problem is first-order accurate. This issue will be revisited later.

3.2. Fröbenius method

Owing that Eq. (6) possesses variable coefficients and a regular singular point at $\eta = 0$, an acceptable analytical solution is the Fröbenius method, a variant of the power series method [8]. Correspondingly, we seek a power series solution of the form

$$\phi(\eta, s) = \eta^s \sum_{k=0}^{\infty} A_k \eta^k \tag{8}$$

where the center of the power series is $\eta = 0$, $A_0 \neq 0$ and s is an arbitrary exponent. Upon introducing Eq. (8) into Eq. (6) the differential-difference equation is satisfied when all the coefficients of the linear independent terms vanish independently. The vanishing of the lower term coefficient supplies the indicial equation, whose solution delivers a repeated indicial exponent $s = 0$. Therefore, the general solution of Eq. (6) at a fixed transversal line ΔX may be expressed as follows

$$\phi(\Delta X, \eta) = \sum_{k=0}^{\infty}(A_k + B_k \ln\eta)\eta^k \tag{9}$$

where the coefficients A_k and B_k satisfy the recurrence relations

$$A_{2k+2} = \frac{1}{\Delta X}\left[\frac{A_k - A_{2k-2}}{(2k+2)^2}\right], \quad k \geqslant 1 \tag{10a}$$

$$B_{2k+2} = 0, \quad k = 0, 1, 2 \ldots \tag{10b}$$

To obtain the particular solution of Eq. (6), the undetermined coefficients A_k and B_k when articulated with the radial boundary conditions of Eq. (7a) and (7b), leads to the pair of relations:

$$\sum_{k=1}^{\infty} kA_k = 1 \tag{11a}$$

and

$$B_k = 0, \quad k = 0, 1, 2, \ldots \tag{11b}$$

Upon specifying a stringent convergence criterion, for instance

$$\left(\frac{A_{2k}\eta^{2k}}{A_0}\right) < 10^{-6} \tag{12}$$

numerous terms have to be included in the computer-extended Fröbenius series of Eq. (9) to achieve the desired accuracy.

4. Discussion of results

A collection of approximate temperature profiles $\phi(\Delta X, \eta)$ of semi-analytical structure in the η-domain $[0, 1]$ has been obtained by evaluating the computer-extended Fröbenius series of Eq. (9) at various axial stations ΔX with a computer code. Abiding by the above convergence criterion, it was found that for small $\Delta X \to 0$ the inclusion of an additional term in the computer-extended Fröbenius series does not affect the value of the dependent variable $\phi(\Delta X, \eta)$ up to eight significant digits. The truncation process for the Fröbenius power series is explained in Table 1 in abridged form. Herein, it may be seen that starting at $\Delta X = 0.0001$ with 162 terms retained, the number of terms diminishes to 11 at $\Delta X = 0.1$ passing through 59 at an intermediate $\Delta X = 0.001$. In addition, it was also found that the number of terms decreases gently as the interval size ΔX gets bigger. The behavior of the Fröbenius series responds directly to the dimension of the embedded stretching parameter ΔX in Eq. (6). In all likelihood, this behavior is consistent with the Graetz/Nusselt series solution of the two-dimensional energy equation (1). The Graetz/Nusselt series converges regularly for large X and diverges severely for short X (Kays and Crawford [9]).

Table 1
Number of terms retained in the computer-extended Fröbenius series of Eq. (9)

ΔX	Number of terms
0.0001	162
0.0002	124
0.0004	88
0.0006	75
0.0008	66
0.001	59
0.002	43
0.004	32
0.006	27
0.008	24
0.01	21
0.02	17
0.04	14
0.06	12
0.08	11
0.1	11
0.2	9
0.4	8
0.6	6
0.8	6
1	6

To assess the goodness of the simple computational procedure that articulates TMOL and the powerful Fröbenius method, it is important to compare the wall temperature results at pre-specified stations ΔX against those wall temperatures evaluated from the classic solutions. Among the classic solutions, one is the Lévêque solution as computed by Bird et al. [5] and the other is the Graetz/Nusselt solution recomputed by Shah [10] almost exactly and reproduced in [1].

Of relative interest, it should be mentioned that the second Lévêque problem is a particularization of the second Graetz/Nusselt problem. The latter refers to the complete thermal development of a viscous fluid flowing laminarly through a round tube with prescribed wall heat flux. The first analytic solutions to the second Graetz/Nusselt problem are attributed to Eagle and Ferguson [11] and Seigel et al. [12].

As a preamble for the discussion, it is worth remembering that the Lévêque solution for prescribed wall temperature is normally acceptable for an upper portion of the thermal entrance region of circular tubes that conform to the reduced interval $0 < x/R \leqslant 0.01RePr$ [1,2]. Incidentally, in the context of the Lévêque solution this inequality may be viewed as the creation of a dimensionless thermal entrance length for tubes heated/cooled with prescribed wall temperature. In practice, this deep region is of significance for short tubes carrying highly viscous fluids in compact heat exchanger tubes [1,2]. Unfortunately, an equivalent inequality for x/R related to uniform wall heat flux in the second Lévêque problem, was not explicitly given in [6]. Nevertheless, the entries in the fourth column of Table 2 may be used for this specific purpose. This avenue suggests that if a relative error of -15% is adopted as the deciding criterion in the context of the first Lévêque problem, the dimensionless thermal entrance length for the second Lévêque problem should be established around $X = 0.01$.

From a fundamental framework for uniform wall heating in round tubes, the dimensionless wall temperature $\phi_w = \phi(X, 1)$ and not the local Nusselt number Nu should be considered as the ultimate target quantity in any comparative study. The wall temperature is a local quantity as opposed to the local Nussel number, which embraces one global and two local thermal quantities (see Eq. (5)). Correspondingly, the wall temperature $\phi_w = \phi(\Delta X, 1)$ from our TMOL-Fröbenius solution will be contrasted in Table 2 against the wall temperature ϕ_w delivered by the two classic solutions, one the Lévêque

Table 2
Comparison of the wall temperature distribution $\phi_w(\Delta X)$

ΔX	Graetz/Nusselt solution [1,10]	Lévêque solution [6]	% Error	TMOL-Fröbenius solution (present work)	% Error
0.001	0.13048	0.12191	−6.57	0.11668	−10.58
0.002	0.16752	0.15360	−8.31	0.14989	−10.52
0.004	0.21627	0.19353	−10.51	0.19373	−10.42
0.006	0.25200	0.22153	−12.09	0.22594	−10.34
0.008	0.28138	0.24383	−13.34	0.25251	−10.26
0.01	0.30689	0.26266	−14.41	0.26564	−10.18
0.02	0.40530	0.33093	−18.35	0.36555	−9.81
0.04	0.54473	0.41694	−23.46	0.49588	−8.97
0.06	0.65531	0.47728	−27.17	0.60277	−8.02
0.08	0.75278	0.52531	−30.22	0.69981	−7.04
0.1	0.84308	0.56588	−32.88	0.79164	−6.10
0.2	1.25716	0.71300	−43.28	1.22021	−2.94
0.4	2.05833	0.89827	−56.36	2.03777	−1.00

solution [6] and the other the Graetz/Nusselt solution [1,10]. It may be observed in the table that the relative errors associated with the Lévêque solution begin with −6.57% at the station $\Delta X = 0.001$ (nearly $X = 0$) and grow monotonically with ΔX as the flow progresses downstream in the tube. Moving down to the station $\Delta X = 0.01$, the relative error increases to −14.41%. The ascending error trend is consistent with the hypothesis of the Lévêque's model simply because the assumptions of linear velocity and "flat plate passage" valid near the tube wall in the vicinity of $X = 0$ breaks down when a threshold value of ΔX is exceeded.

On the other hand, our ϕ_w results via the TMOL-Fröbenius solution, exhibit intrinsic relative errors that start with a value of −10.58% at $\Delta X = 0.001$ (practically $X = 0$) and at a distant $\Delta X = 0.01$ the relative error is −10.18%. Despite that the relative errors decrease slightly with ΔX, it may be confirmed that the error band remains stable between $\Delta X = 0.001$ and 0.01. It is curious that the relative errors produced by the two techniques level off at the station $\Delta X = 0.004$. Beyond $\Delta X = 0.01$, the ϕ_w error descends markedly as the viscous flow continues to move into the far downstream region.

The thermal entrance region for round tubes with isoflux heating has been tacitly established at $\Delta X = 0.1$ [1,10]. For obvious reasons, the end station of the thermal entrance region deserves special attention. At this borderline location $\Delta X = 0.1$, the ϕ_w error delivered by our TMOL-Fröbenius solution diminishes to −6.1%. In fact, this is indeed a surprising finding. As a point of reference, at this same location $\Delta X = 0.1$ the error for ϕ_w supplied by the Lévêque solution grows enormously to −32.88%.

We also included in Table 2, the computed wall temperatures ϕ_w for two additional stations $\Delta X = 0.2$ and 0.4 that lie well outside the confines of the thermal entrance region. It is interesting to see that at these two far-away stations, the ϕ_w errors by the TMOL-Fröbenius solution continue to diminish steadily with increments in ΔX, reaching values of −2.94% at $\Delta X = 0.2$ and −1.00% at $\Delta X = 0.4$. Also, the correctness of this behavior at stations far away from $X = 0$ is an astonishing discovery. As opposed to this, the ϕ_w errors by the Lévêque solution keep increasing steadily with increments in ΔX, reaching values of −43.28% at $\Delta X = 0.2$ and −56.36% at $\Delta X = 0.4$.

For completeness, results for the local Nusselt number Nu are also reported in this work. It may be seen in Fig. 1 that the asymptotic local Nusselt number sub-distribution $Nu_{x\to 0}$ produced by approximate TMOL/Fröbenius method slightly overpredicts the exact local Nusselt number distribution Nu coming from the exact Graetz series [1,10]. A peculiarity of the two curves is that they are parallel for $0.001 < \Delta X < 0.1$. Meanwhile, the asymptotic local Nusselt number sub-distribution $Nu_{x\to 0}$ that emerged from the approximate Lévêque solution coincides with the exact local Nusselt number

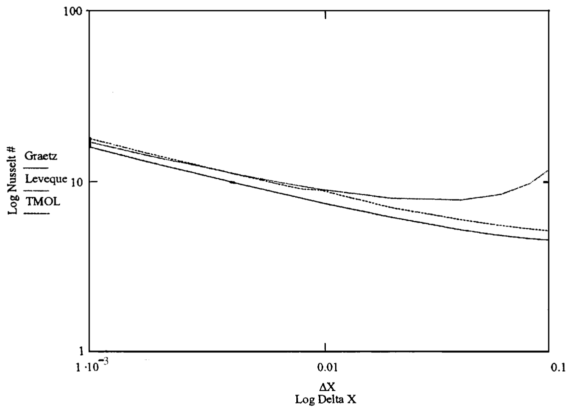

Fig. 1. Comparison of the local Nusselt number in the thermal entrance region.

distribution Nu up to $\Delta X = 0.01$. Thereafter, the Nu associated with the Lévêque solution curves up with ΔX and deviates from the exact Nu.

The success of TMOL/Fröbenius method in predicting wall tube temperatures with relative precision may be explained as follows. The wall temperature curve consists of two parts, a first concave curved line and a second positive-sloped straight line. In the first part, the wall temperature curves up from 0 to 0.1 in the region of thermal development. In this particular region, the errors of the backward formulation for the axial temperature derivative being proportional to ΔX stay within a 10% on a consistent basis. At the threshold station $\Delta X = 0.1$, the wall temperature turns into a positive-sloped straight line possessing an error of 6.1%. Thereafter, the errors keep decreasing gradually until reaching 1% at $\Delta X = 0.4$. Based on this amazing tendency, it may be inferred that the errors diminish because the slope of the backward formulation approaches the slope of the inclined positive-sloped straight line (see Appendix B).

5. Conclusions

The accuracy delivered by the TMOL/Fröbenius method for the study of isoflux heating of round tubes carrying laminar fluid flows is an irrefutable proof of its superiority with respect to the Lévêque method implemented by Bird et al. [6], who followed the footsteps of Lévêque [4]. The TMOL/Fröbenius method can be utilized with confidence for calculating wall tube temperatures not only in the reduced thermally developing region in the context of Lévêque, but in the entire thermal developing region in the general sense of Graetz/Nusselt. Qualitatively speaking, the Lévêque method can be safely employed inside the abridged thermal developing region up to $\Delta X = 0.01$. In contrast, the TMOL/Fröbenius method applies to the whole thermal developing region and beyond, i.e., for any value of ΔX, smaller than 0.01 or greater than 0.01. This high level of quality is in fact a pleasant discovery that the outcome of this paper has aptly unveiled.

Appendix A. Determination of the mean bulk temperature in a TMOL environment

The differential-difference equation (6) may be rewritten as

$$\frac{1}{\eta}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta\frac{\mathrm{d}\phi}{\mathrm{d}\eta}\right) - \frac{1}{\Delta X}(1-\eta^2)\phi = 0 \quad \text{(A.1)}$$

Multiplying this equation by $\eta\,\mathrm{d}\eta$, it becomes

$$\mathrm{d}\left(\eta\frac{\mathrm{d}\phi}{\mathrm{d}\eta}\right) - \frac{1}{\Delta X}(1-\eta^2)\phi\eta\,\mathrm{d}\eta = 0 \quad \text{(A.2)}$$

Integration of this equation between the limits 0 and 1 gives

$$\int_0^1 \mathrm{d}\left(\eta\frac{\mathrm{d}\phi}{\mathrm{d}\eta}\right) - \frac{1}{\Delta X}\int_0^1 (1-\eta^2)\phi\eta\,\mathrm{d}\eta = 0 \quad \text{(A.3)}$$

Next, performing the integration results in

$$\left(\eta\frac{\mathrm{d}\phi}{\mathrm{d}\eta}\right)_{\eta=1} - \left(\eta\frac{\mathrm{d}\phi}{\mathrm{d}\eta}\right)_{\eta=0} - \frac{1}{4\Delta X}\left[4\int_0^1 (1-\eta^2)\phi\eta\,\mathrm{d}\eta\right] = 0 \quad \text{(A.4)}$$

In concordance with the boundary conditions of Eq. (7a) and (7b), the first term becomes one while the second term vanishes. On the other hand, the expression in brackets in the third term is recognized as the mean bulk temperature ϕ_b as given by Eq. (3). This combination of factors yields

$$1 - 0 - \frac{1}{4\Delta X}(\phi_b) = 0 \quad \text{(A.5)}$$

This paves the way for the following linear variation

$$\phi_b = 4\Delta X \quad \text{(A.6)}$$

which is represented by Eq. (4).

Appendix B. Implications of the TMOL-Fröbenius method

The formal definition of the backward finite-difference formula is [8]:

$$f'(x_0) = \frac{f(x_0) - f(x_0 - \Delta x)}{\Delta x} - \frac{\Delta x}{2}f''(\xi) \quad \text{(B.1)}$$

For small values of Δx, the difference quotient can be used to approximate $f'(x_0)$ with an error bounded by $M\Delta x/2$, if M is a bounded quantity for $x \in [a,b]$.

An important consideration in numerical differentiation is the effect of the round-off error. The truncation portion of the error in a numerical differentiation technique will decrease if the interval Δx is reduced, but at the expense of increased round-off portion of the error. In practice, it is difficult to separate both errors and compute an optimal interval ΔX_{opt} to be used in approximating the first derivative, since we have no knowledge of the second derivative of the function $f''(\xi)$ for $x \in [a,b]$. To complicate things, the interval $[a,b]$ is not kept constant in the present computational technique.

The issue to be addressed here is concerned with the estimation of the maximum interval ΔX_{max} in Eq. (8), which provides temperature results of comparable quality to those given by the Lévêque solution. An intituitive discussion will revolve around the approximation of the slope of the wall temperature curve by a secant in the vicinity of X tending to zero. This is done in conformity with the definition of the backward formula in Eq. (B.1) on a global basis.

References

[1] R. Shah, A.L. London, Laminar Flow Forced Convection in Ducts, Academic, New York, 1978.
[2] V. Gnielinski (Ed.), Wärmeübertragung Bei Der Strömung Durch Rohre, VDI-Wärmmeatlas, VDI-Verlag, Düsseldorf, Germany, 1984.
[3] G.F. Hewitt et al. (Eds.), International Encyclopedia for Heat and Mass Transfer, CRC Press, Boca Raton, FL, 1997.
[4] M.A. Lévêque, Les lois de la transmission de chaleur par convection, Ann. Mines Memories (12–13) (1928), pp. 201–299, 305–362, 381–415.
[5] J. Kevorkian, Partial Differential Equations, Wadsworth and Brooks/Cole, Pacific Grove, CA, 1990, pp. 493–494.
[6] R.B. Bird, W.E. Steward, E.N. Lightfoot, Transport Phenomena, John Wiley, New York, 1960, pp. 363–364.
[7] E. Rothe, Zweidimensionale parabolische Randwertaufgaben als Grenzfall eindimensionaler Randwertaufgaben, Math. Ann. 102 (1930) 650–670.
[8] C.M. Bender, S.A. Orszag, Advanced Mathematical Methods for Scientists and Engineers, McGraw-Hill, New York, 1978.
[9] W.M. Kays, M.E. Crawford, Convective Heat and Mass Transfer, third ed., McGrawHill, New York, 1993.
[10] R. Shah, Thermal entry length solutions for the circular tube and parallel-plate channels, Paper No. HMT-11–75, Proceedings of the National Heat and Mass Transfer Conference (India), Indian Institute of Technology, Bombay, India, 1975.
[11] A. Eagle, R.M. Ferguson, On the coefficient of heat transfer from the internal surface of tube walls, Proc. Roy. Soc. (London) A 127 (1930) 540–555.
[12] R. Seigel, E.M. Sparrow, T.M. Halman, Steady laminar heat transfer in a circular tube with prescribed wall heat flux, Appl. Sci. Res., Ser. A 7 (1958) 386–392.